A handy approximation for the error function and its inverse

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February 6, 2008

The error function

$$\operatorname{erf} x \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx \tag{1}$$

can be numerically approximated in many ways. In some applications it is useful if not only erf$x$, but also its inverse function, $\operatorname{erf}^{-1} x$, defined by

$$x = \frac{2}{\sqrt{\pi}} \int_0^{\operatorname{erf}^{-1} x} e^{t^2} dt = \operatorname{erf}\left(\operatorname{erf}^{-1} x\right), \tag{2}$$

is approximated by a simple formula. The following approximation is then useful,

$$\operatorname{erf} x \approx \left[1 - \exp\left(-x^2 \frac{\frac{4}{\pi} + ax^2}{1 + ax^2}\right)\right]^{1/2}, \tag{3}$$

where the constant a is

$$a \equiv \frac{8}{3\pi} \frac{\pi - 3}{4 - \pi} \approx \frac{8887}{63473} \approx \frac{7}{50} = 0.14. \tag{4}$$

This formula can be derived by the method explained in Ref. [1]. One writes an ansatz of the form

$$\operatorname{erf} x \approx \left[1 - \exp\left(-\frac{P(x)}{Q(x)}\right)\right]^{1/2}, \tag{5}$$

where $P(x)$ and $Q(x)$ are unknown polynomials, and finds P, Q such that the known series expansion of erf x at $x = 0$ and the asymptotic expansion at $x = +\infty$ are reproduced within first few terms. In this way one obtains explicit coefficients of the polynomials P and Q to any desired degree.

A numerical evaluation shows that Eq. (3) provides an approximation for erf x correct to better than $4 \cdot 10^{-4}$ in relative precision, *uniformly* for all real $x \geq 0$, as illustrated in Fig. 1. For negative x one can use the identity

$$\operatorname{erf}(-x) = -\operatorname{erf} x. \tag{6}$$

The function in Eq. (3) has the advantage that it can be easily inverted analytically,

$$\operatorname{erf}^{-1} x \approx \left[-\frac{2}{\pi a} - \frac{\ln\left(1 - x^2\right)}{2} + \sqrt{\left(\frac{2}{\pi a} + \frac{\ln\left(1 - x^2\right)}{2}\right)^2 - \frac{1}{a}\ln\left(1 - x^2\right)}\right]^{1/2}. \tag{7}$$

The relative precision of this approximation is better than $4 \cdot 10^{-3}$, uniformly for all real x in the interval $(0, 1)$, as illustrated in Fig. 2.

An improvement of the precision is possible if one chooses the coefficient $a \approx 0.147$ instead of the above value $a \approx 0.140$. With $a \approx 0.147$ the largest relative error of Eq. (3) is about $1.3 \cdot 10^{-4}$, and the relative error of Eq. (7) is about $2 \cdot 10^{-3}$.

One application of the formula (7) is to Gaussian statistics. Suppose we would like to test that a large data sample $x_1, ..., x_N$ comes from the normal distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{8}$$

We can split the total range of x into bins with boundaries $b_1, b_2, ...$ and count the number of sample points that fall into each bin, $b_i < x < b_{i+1}$. If $\{x_i\}$ come from the Gaussian distribution, then the expected number of samples within the bin $b_i < x < b_{i+1}$ should be

$$n(b_i, b_{i+1}) = N \int_{b_i}^{b_{i+1}} p(x)dx = \frac{N}{2}\left(\mathrm{erf}\frac{b_{i+1}}{\sigma\sqrt{2}} - \mathrm{erf}\frac{b_i}{\sigma\sqrt{2}}\right). \tag{9}$$

To have better statistics, one should choose the bin boundaries so that the expected numbers of samples are equal in all bins. This is achieved if the bin boundaries b_i are chosen according to the formula

$$b_i = \sigma\sqrt{2}\mathrm{erf}^{-1}\frac{2i - B - 1}{B + 1}, \quad i = 1, ..., B, \tag{10}$$

where $(B + 1)$ is the total number of bins and erf^{-1} is the inverse function to $\mathrm{erf}\, x$. Using Eq. (7) it is straightforward to compute the bin boundaries b_i.

The plots were produced by the following Maple code:

```
a:=0.14; # or other value
f1:=x->sqrt(1-exp(-x^2*(4/Pi+a*x^2) / (1+a*x^2)));
plot([1-erf(x)/f1(x) ], x=0..4);
f2:=x->sqrt(-(2/Pi/a+ln(1-x^2)/2) + sqrt(-ln(1-x^2)/a
+(2/Pi/a+ln(1-x^2)/2)^2) );
inverf:=x->solve(erf(y)=x,y);
plot(1-f2(x)/inverf(x), x=0.01 ..1);
```

The author thanks David W. Cantrell who discussed the shortcomings of a previous version of this note and suggested $a = 0.147$ in a Google Group online forum *sci.math* in December 2007.

References

[1] S. Winitzki, *Uniform approximations for transcendental functions*, in Proc. ICCSA-2003, LNCS 2667/2003, p. 962.

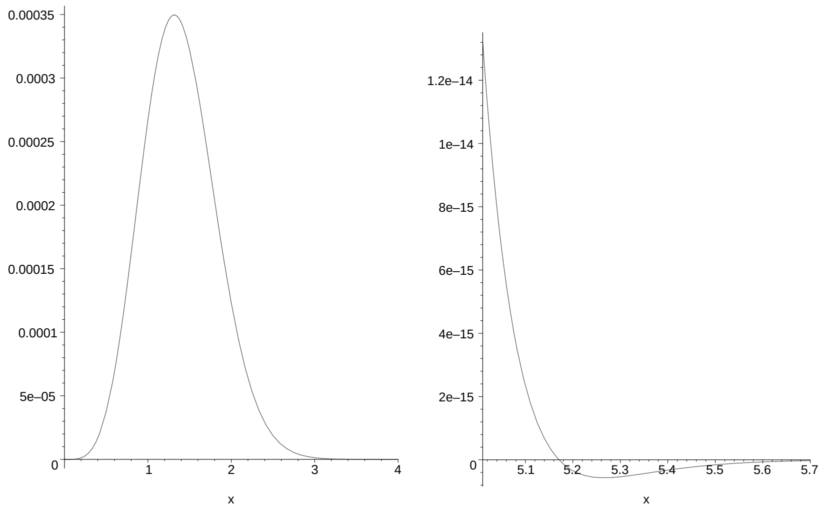

Figure 1: Relative error of the approximation (3) is below 0.00035 for all $x > 0$. Note that the relative error is not everywhere positive; the plot at right shows that $\operatorname{erf} x$ is underestimated at large x.

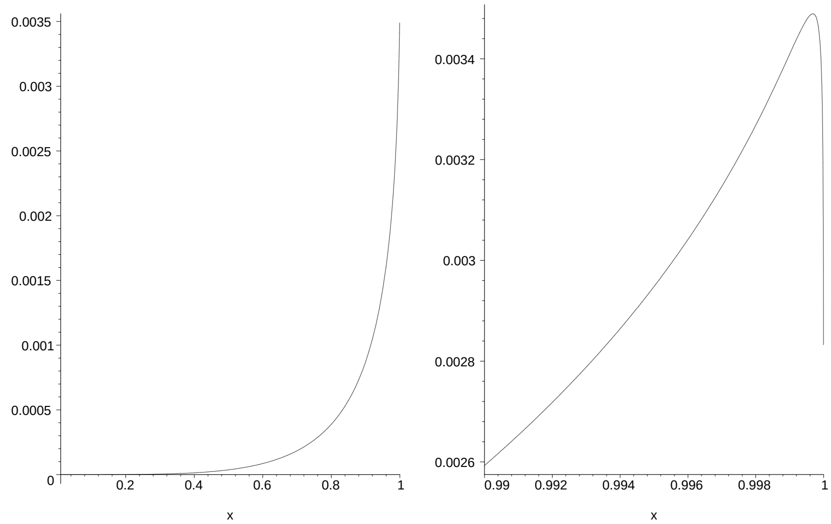

Figure 2: Relative error of the approximation (7) is below 0.0035 for all $x > 0$. The plot at right shows that the error is largest near $x = 1$.